Coordinate Changes for Integrals

In Calc I we solved things like

$$\int_0^5 xe^{x^2}\,dx \quad \begin{cases} u = x^2 \\ du = 2x\,dx \end{cases} \leftarrow \text{parameter change}$$

In double integrals, we made a polar coordinate change

$$dA_{cart} = r\,dA_{polar}$$

Q: How do we deal with more general changes?
A: We use the Jacobian to understand diff. changes

Def: Suppose $\begin{cases} x_1 = x_1(u_1, u_2, \ldots, u_n) \\ x_2 = x_2(u_1, u_2, \ldots, u_n) \\ \quad\vdots \\ x_n = x_n(u_1, u_2, \ldots, u_n) \end{cases}$ is a coordinate change by diff functions. The Jacobian of the coordinate change is

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} = \det\begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

Ex: Compute the signed Jacobian of Polar transform $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

Sol: $\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin(\theta) \\ \sin\theta & r\cos(\theta) \end{bmatrix}$

$= \cos\theta\, r\cos\theta - (-r\sin\theta\sin\theta) = r(\cos^2\theta - \sin^2\theta) = r$

NB: swapping the order

$\frac{\partial(x,y)}{\partial(\theta,r)} = \det\begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix}$

$= -r\sin^2\theta - r\cos^2(\theta) = -r$

The unsigned Jacobian is $\left|\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}\right|$

11)

Prop: If $f(x_1, x_2, \dots, x_n)$ is a cts function and

$$\begin{cases} x_1 = x_1(u_1, u_2, \dots, u_n) \\ x_2 = x_2(u_1, u_2, \dots, u_n) \\ x_n = x_n(u_1, u_2, \dots, u_n) \end{cases}$$

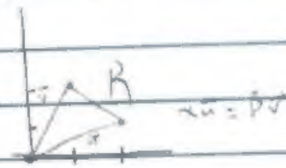
is a diff coord trans

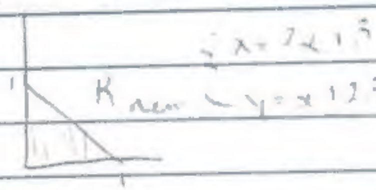
$$\int_{R_{old}} f \, dV_{old} = \int_{R_{old}} f(x_1(u_1, \dots, u_n), \dots, x_n(u_1, \dots, u_n)) \left| \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} \right|$$

NB: This matches with our work for polar coords

Example: Compute $\iint_R (x-3y)\,dA$ for R the triangle w/ vertices $(0,0), (1,2), (2,1)$

Sol: is the parameterization when $(\alpha, \beta) = (1,0)$ and $(x(\alpha,\beta), y(\alpha,\beta)) = (2,1)$ and when $(\alpha,\beta) = (0,1)$ we have $(x(\alpha,\beta), y(\alpha,\beta)) = (1,2)$ and when $(\alpha,\beta) = (0,0)$ yields $(x(\alpha,\beta), y(\alpha,\beta)) = (0,0)$





By high school geometry, this Linear change takes R_{new} to R_{old}

$R_{new} = \{(\alpha,\beta) : 0 \le \alpha \le 1, 0 \le \beta \le 1-\alpha\}$

The Jacobian is $\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \det \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{bmatrix} = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$

$$\therefore \iint_{R_{old}} (x-3y)\, dA_{old} = \iint_{R_{new}} \left((2\alpha+\beta) - 3(\alpha+2\beta) \right) \left| \frac{\partial(x,y)}{\partial(\alpha,\beta)} \right| dA_{new}$$

$$= \int_{\alpha=0}^{1} \int_{\beta=0}^{1-\alpha} (-\alpha - 5\beta)\, 3 \, d\beta \, d\alpha$$

$$= -3 \int_0^1 \int_0^{1-\alpha} (\alpha - 5\beta)\, d\beta \, d\alpha$$

$$= -3 \int_0^1 \left[\alpha\beta - \tfrac{5}{2}\beta^2 \right]_0^{1-\alpha}$$

$$= -3 \int_0^1 (1-\alpha)\left(\alpha + \tfrac{5}{2}(1-\alpha)\right) d\alpha$$

$$= -\frac{3}{2} \left[5\alpha - 4\alpha^2 + \alpha^3 \right]_0^1 = -3$$

Now lets generalize polar coords in 3-space

I. Naive way. Cylindrical coords

IDEA: parameterize the plane by polar coords
leave the orthogonal axis unchanged

Ex: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

Jacobian is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \cos\theta(r\cos\theta \cdot 1 - 0) - (-r\sin\theta)(\sin\theta \cdot 1 - 0) + 0(\sin\theta \cdot 0 - \ldots)$

$= r\cos^2\theta + r\sin^2\theta = r$

$\therefore$ when we compute an integral in cylindrical coords, we need to multiply the differential by r

* This is true of all cylindrical changes

Ex: compute $\iiint_R (x+y+z)\,dV$ for the solid with first octant and below paraboloid $4-x^2-y^2=z$

Sol: In coords $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$

$\Rightarrow R_{cyl} = \{(r,\theta,z) ; 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2,\ 0 \le z \le 4-r^2\}$

$\therefore \iiint_R (x+y+z)\,dA_{cyl} = \int_{r=0}^{2} \int_{z=0}^{4-r^2} \int_{\theta=0}^{\pi/2} (r\cos(\theta) + r\sin(\theta) + z)\, r \, d\theta\, dz\, dr$

$= \int_0^2 \int_0^{4-r^2} r\left[r\sin\theta - r\cos\theta + z\theta\right]_0^{\pi/2} dz\, dr$

$\int_0^2 \int_0^{4-r^2} r\left((r - 0 + z\tfrac{\pi}{2}) - (0 - r + 0)\right) dz\, dr$

$= \int_0^2 \int_0^{4-r^2} \left(2r^2 + \frac{\pi}{2} zr\right) dz\, dr$

$= \int_0^2 \left[2r^2 z + \frac{\pi}{4} z^2 r\right]_{z=0}^{4-r^2} dr$

$= \int_0^2 \left(2r^2(4-r^2) + \frac{\pi}{4} r(4-r^2)^2\right) dr$

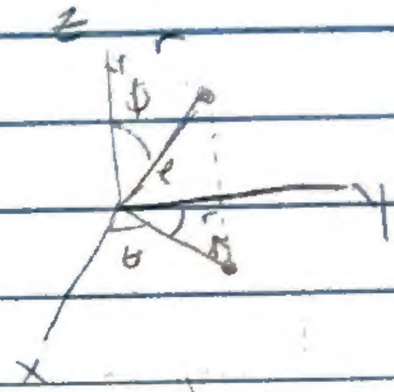
$= \int_0^2 8r^2 - 2r^4 + \frac{\pi}{4}(16r - 8r^3 + r^5)\, dr$

$= \left[\frac{8}{3}r^3 - \frac{2}{5}r^5 + \frac{\pi}{4}\left(8r^2 - 2r^4 + \frac{1}{6}r^6\right)\right]_0^2$

$= \frac{64}{3} - \frac{32}{5} + \frac{\pi}{4}\left(32 - 32 + \frac{32}{3}\right) - 0$

$= \frac{64}{3} - \frac{32}{5} + \frac{8\pi}{3}$

II: Sphereical Coordinates



In sphereical coordinates, we parameterize points (x,y,z) using 3 pieces of data

ρ = distance from origin

θ = angle made w/ $+x$-axis and point $(x,y,0)$

ϕ = angle made w/ $+z$-axis and point (x,y,z)

Note: $\sin(\phi) = \frac{r}{\rho}$, so $r = \rho\sin(\phi)$

$\therefore$ in our parameterization

$$\begin{cases} x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta) \\ y = r\cos(\theta) = \rho\sin(\phi)\sin(\theta) \end{cases}$$

Also, $\cos(\phi) = \frac{z}{\rho}$, so $z = \rho\cos(\phi)$

Hence the sphereical coord parameterization is

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$$